Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.32, No. 2, 145-147

An inequality of Clifford indices for a finite covering of curves

Edoardo Ballico[1] and Masaaki Homma[2]

Abstract. We prove that for a finite covering of curves the Clifford index of the source is at least that of the target.

Keywords: Clifford index, finite covering of curves.

Introduction

The *Clifford index* c_X of a smooth curve X is, by definition, the smallest possible value of the expression

$$\deg D - 2h^0(X, \mathcal{O}_X(D)) + 2$$

for a divisor D with $h^0(X, \mathcal{O}_X(D)) \geq 2$ and $h^1(X, \mathcal{O}_X(D)) \geq 2$. The notion was introduced by H. H. Martens [M], and has been studied by a number of authors from various points of view.

Let $f: X \to Y$ be a finite covering of smooth curves over an algebraically closed field. It seems natural to expect that $c_X \geq c_Y$. In this note, we prove it. Our proof is based on a result of Coppens and Martens [CM, Cor. 3.2.5], where the ground field is the complex numbers. So our proof works only in characteristic 0.

The Clifford index makes sense only when X is of genus g_X at least 4 or is hyperelliptic with $g_X = 3$. When Y is hyperelliptic, the inequality $c_X \geq c_Y$ is clear because $c_Y = 0$. So we may assume $g_X \geq 4$ and $g_Y \geq 4$.

Received 30 November 2000.
[1]Partially supported by MURST (Italy).
[2]Partially supported by JSPS.

Theorem. *For a finite covering $f\colon X \to Y$ of smooth curves whose genera are at least 4 over an algebraically closed field of characteristic 0, we have $c_X \geq c_Y$.*

Proof. Choose a linear system g^r_d on X which computes the Clifford index c_X of X. Hence $d = c_X + 2r$. For a canonical divisor K, the linear system $|K - g^r_d|$ also computes the Clifford index of X. Hence we may assume that $d \leq g_X - 1$. Let us consider the complete linear system $|f_*D|$ for $D \in g^r_d$. Since $\dim |f_*D| \geq r$ and $\deg |f_*D| = d$, the proof is done if $h^1(Y, \mathcal{O}_Y(f_*D)) \geq 2$. When X is either hyperelliptic or trigonal, the inequality $h^1(Y, \mathcal{O}_Y(f_*D)) \geq 2$ holds by the Riemann–Roch theorem because the genus g_Y of Y is at least 4.

Now we consider the case when X is bi-elliptic, whose Clifford index is 2 and computed by a pencil g^1_4. If $g_Y \geq 5$, then we have $h^1(Y, \mathcal{O}_Y(f_*D)) \geq 2$, and get the inequality $c_X \geq c_Y$. If $g_Y = 4$, then Y is trigonal, and so $c_Y = 1$, which means that the inequality $c_X \geq c_Y$ is true.

Next we handle the case where X is a plane quintic curve, which is the only remaining case for $c_X \leq 1$. (For the classification of curves X with $c_X = 1$, see [M, (2.51)].) Since X is of genus 6, we have $g_Y \leq 3$ by the Hurwitz formula, which is out of our consideration.

Thus we may assume that $c_X \geq 2$ and X is not bi-elliptic. First we assume that $g_X > 2c_X + 5$, where g_X is the genus of X. Then by [CM, Cor. 3.2.5], we have

$$d \leq 3c_X/2 + 3. \tag{1}$$

Suppose that $h^1(Y, \mathcal{O}_Y(f_*D)) \leq 1$. Then we have

$$\dim |f_*D| \leq d - g_Y + 1$$

by the Riemann–Roch theorem. Hence we have

$$g_Y - 1 \leq (c_X + d)/2 \tag{2}$$

because $\dim |f_*D| \geq r$ and $d = c_X + 2r$. Recall that $c_Y \leq (g_Y - 1)/2$ by the existence theorem of Brill–Noether theory (see for example [ACGH, p. 206]). Therefore,

$$\begin{aligned} c_Y \leq (g_Y - 1)/2 &\leq (c_X + d)/4 && \text{by (2)} \\ &\leq 5c_X/8 + 3/4 && \text{by (1)} \\ &\leq c_X && \text{because } c_X \geq 2. \end{aligned}$$

Finally we consider the remaining case, that is, $g_X \leq 2c_X + 5$. By the Hurwitz formula, we have $g_X - 1 \geq 2(g_Y - 1)$. So we have

$$\begin{aligned} c_Y \leq (g_Y - 1)/2 &\leq (g_X - 1)/4 \\ &\leq c_X/2 + 1 \qquad \text{by our assumption} \\ &\leq c_X \qquad \text{because } c_X \geq 2. \end{aligned}$$

The proof is now complete.

Acknowledgment. This work was done while the second author was visiting University of Trento. He is deeply grateful to University of Trento for their hospitality.

References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of algebraic curves* Vol. 1, Grundlehren Math. Wiss. **267:** (1985), Springer-Verlag.

[CM] M. Coppens and G. Martens, *Secant spaces and Clifford's theorem*, Compositio Math. **78:** (1991), 193–212.

[M] H. H. Martens, *Varieties of special divisors on a curve* II, J. Reine Angew Math. **233:** (1968), 89–100.

Edoardo Ballico
Department of Mathematics
University of Trento
38050 Povo (TN)
Italy

E-mail: ballico@science.unitn.it

Masaaki Homma
Department of Mathematics
Kanagawa University
Yokohama 221-8686
Japan

E-mail: homma@cc.kanagawa-u.ac.jp